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The Generalizes of G^{**}-Autonilpotent Groups

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Abstract

We introduced a series on the subgroups generated by a group G and IA(G) and defined G^{**}-autonilpotency on this series [2]. In this paper, we generalize these concepts and then study some properties of them and their relationships.

Keywords: IA-group, n-IA-commutator series, autonilpotent groups, G^{**}-autonilpotent groups, n-IA-nilpotent groups.

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1. Introduction

Let G be a group. Let us denote by Aut(G) and G', respectively, the full automorphism group and the commutator subgroup. Bachmuth [1] in 1965 defined an IA-automorphism of a group G as

$$IA(G) = \left\{\alpha \in Aut(G) \mid g^{-1}\alpha(g) = [g, \alpha] \in G', \ \forall\, g \in G\right\}.$$

Hegarty [4] in 1994 introduced the autocommutator subgroup as follows:

$$K(G) = \langle [g, \alpha] \mid g \in G,\ \alpha \in Aut(G)\rangle.$$

On the similar lines, Ghumde and Ghate [3] in 2015 introduced the subgroup

$$G^{**} = \langle [g, \alpha] \mid g \in G,\ \alpha \in IA(G)\rangle.$$

For any group G, $G' = G^{**} \leqslant K(G)$.

First, we study the conditions in which G^{**} is equal to K(G). The following proposition clearly states these conditions.

Proposition 1.1. For a group G, $G^{**} = K(G)$ if one of the following conditions holds

1) G be a complete group, i.e., $G = G'$.
2) G' be of index two of G, because then IA(G)=Aut(G).
3) Inn(G)=Aut(G).

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For any group G, $G^{**} = \langle 1 \rangle$ if and only if G is a trivial or abelian group. Also, G^{**} is an abelian group if and only if G is a metabelian group.

Lemma 1.2. Let $G = H_1 \times H_2$, $H_1 \neq \langle 1 \rangle$ and $H_1 \cap G^{**} = \langle 1 \rangle$, then $H_1 \cong C_2$.

Proof. Suppose by way of contradiction that $|H_1| > 2$, then H_1 has a nontrivial automorphism α and we can extend α to an automorphism β of G where $\beta(h_1) = \alpha(h_1)$, for all $h_1 \in H_1$ and $\beta(h_2) = h_2$, for all $h_2 \in H_2$. If $h_1 \in H_1$ is arbitrary, then

$$[h_1, \beta] = [h_1, \alpha] \in H_1 \cap G^{**} = \langle 1 \rangle.$$

Thus $\alpha(h_1) = h_1$ contradicting the fact that α is nontrivial. □

Parvaneh and Moghaddam [6] in 2010 introduced the concept of autonilpotent groups. They defined the autocommutator subgroup of weight m+1 as

$$\begin{aligned} K_m(G) &= [K_{m-1}(G), Aut(G)] \\ &= \langle [g, \alpha_1, \alpha_2, \ldots, \alpha_m] \mid g \in G,\ \alpha_1, \alpha_2, \ldots, \alpha_m \in Aut(G) \rangle, \end{aligned}$$

for all $m \geqslant 1$, and obtained a descending chain of autocommutator subgroups of G as follows:

$$\cdots \subseteq K_m(G) \subseteq \cdots \subseteq K_2(G) \subseteq K_1(G) = K(G) \subseteq K_0 = G.$$

Also, they called a group to be autonilpotent of class at most m if $K_m(G) = G$, for some positive integer m.

Mohebian and Hosseini [5] in their paper generalized these concepts and defined the n-autocommutator subgroup inductively as follows:

$$\begin{aligned} K_0^n(G) =& G, \\ K_1^n(G) =& K^n(G) = \langle [g, \alpha^n] \mid g \in G,\ \alpha \in Aut(G) \rangle, \\ \text{and for } n \geqslant 2:\ K_m^n(G) =& \langle [g, \alpha_1^n, \ldots, \alpha_m^n] \mid g \in G,\ \alpha_1, \ldots, \alpha_m \in Aut(G) \rangle. \end{aligned}$$

For any group G, $K_m^n(G) \overset{ch}{\leqslant} G$. Also, they introduced the lower n-autoceneral series of G as

$$\cdots \subseteq K_m^n(G) \subseteq \cdots \subseteq K_2^n(G) \subseteq K_1^n(G) = K^n(G) \subseteq K_0^n(G) = G$$

and they called a group G is an n-autonilpotent group if the lower n-autoceneral series finally terminate in the identity subgroup. In special case that $n = 1$, the n-autonilpotent groups are the same autonilpotent groups.

We [2] defined the IA-commutator series or G^{**} series of G as

$$\cdots \subseteq G_m^{**} \subseteq \cdots \subseteq G_2^{**} \subseteq G_1^{**} = G^{**} = G' \subseteq G_0^{**} = G \tag{1.1}$$

where m is a positive integer and

$$\begin{aligned} G_m^{**} =& \langle [g, \alpha_1, \ldots, \alpha_m] \mid g \in G,\ \alpha_1, \ldots, \alpha_m \in IA(G) \rangle \\ =& [G_{m-1}^{**}, IA(G)]. \end{aligned}$$

$G_m^{**} \leqslant K(G)$ and if G is an abelian group, then $G_m^{**} = \langle 1 \rangle$, for every positive integer m. A group G is called G^{**}-autonilpotent group of class at most m if the series (1.1) finally terminate in the identity subgroup. The autonilpotent groups are G^{**}-autonilpotent, but we provide an example in which the converse is not generally valid. All of the abelian groups are G^{**}-autonilpotent, but

$$G = \bigoplus_{i=1}^{l} \mathbb{Z}_{2^{m_i}}, \qquad l > 1,\ \ m_1 = m_2 \geqslant \cdots \geqslant m_l$$

is not autonilpotent, because $K_n(G) = G$.

On the similar lines, we generalize these concepts and then study their properties.

Definition 1.3. For each positive integer m and n, we define

$$G_m^{n*} = \langle [g, \alpha_1^n, \ldots, \alpha_m^n] \mid g \in G,\ \alpha_1, \ldots, \alpha_m \in IA(G) \rangle.$$

Thus, we have the n-IA-commutator series as

$$\cdots \subseteq G_m^{n*} \subseteq \cdots \subseteq G_2^{n*} \subseteq G_1^{n*} = G^{n*} \subseteq G_0^{n*} = G. \tag{1.2}$$

2. Properties of the terms of the n-IA-commutator series

Proposition 2.1. Let G be a group, then for every positive integer m and n, G_m^{n*} is a characteristic subgroup of G.

Proof. Obviously, $G_m^{n*} \leqslant G$. Now, let $\beta \in Aut(G)$ and $[g, \alpha_1^n, \ldots, \alpha_m^n] \in G_m^{n*}$, for any $g \in G$ and $\alpha_1, \ldots, \alpha_n \in IA(G)$. Then, one can write

$$\begin{aligned}
\beta^n([g, \alpha_1^n, \ldots, \alpha_m^n]) &= \beta^n\big(g^{-1}\alpha_1^n \cdots \alpha_m^n(g)\big) \\
&= \beta^n(g^{-1})\beta^n\big(\alpha_1^n \cdots \alpha_m^n(g)\big) \\
&= \big(\beta(g)^{-n}\big)\beta^n\big(\alpha_1^n \cdots \alpha_m^n\beta^{-n}\beta^n(g)\big) \\
&= \big(\beta(g)^{-n}\big)\beta^n\alpha_1^n \cdots \alpha_m^n\beta^{-n}\big(\beta^n(g)\big) \\
&= [\underbrace{\beta(g)^{-n}}_{\in G}, \underbrace{\beta^n\alpha_1^n \cdots \alpha_m^n\beta^{-n}}_{\in IA(G) \trianglelefteq Aut(G)}] \in G_{m+2}^{n*} \leqslant G_m^{n*}.
\end{aligned}$$

□

Lemma 2.2. a) Let G and H be two arbitrary groups, then for any positive integer m and n,

$$G_m^{n*} \times H_m^{n*} \subseteq (G \times H)_m^{n*}.$$

b) If G and H are finite groups such that $(|G|, |H|) = 1$, Then for any positive integer m and n,

$$G_m^{n*} \times H_m^{n*} = (G \times H)_m^{n*}.$$

Proof. a) Because $IA(G \times H) = IA(G) \times IA(H)$ and

$$G^{**} \times H^{**} = G' \times H' = (G \times H)' = (G \times H)^{**},$$

it follows by induction on m that

$$([g, \alpha_1^n, \ldots, \alpha_m^n], [h, \beta_1^n, \ldots, \beta_m^n]) = [(g, h), \alpha_1^n \times \beta_1^n, \ldots, \alpha_m^n \times \beta_m^n],$$

for $\alpha_i \in IA(G)$, $\beta_i \in IA(H)$, $g \in G$ and $h \in H$.

b) We prove that

$$(G \times H)_m^{n*} \subseteq G_m^{n*} \times H_m^{n*}$$

which give sufficient condition for the result. It is not difficult to see that $\sigma|_G \in IA(G)$ and $\sigma|_H \in IA(H)$ for all $\sigma \in IA(G \times H)$. Now, by induction on m and n, we have

$$[(g, h), \sigma_1^n, \ldots, \sigma_m^n] = ([g, \sigma_1^n|_G, \ldots, \sigma_m^n|_G], [h, \sigma_1^n|_H, \ldots, \sigma_m^n|_H]),$$

for every $g \in G$, $h \in H$ and $\sigma_1, \ldots, \sigma_m \in IA(G \times H)$. This implies the result. □

Lemma 2.3. Let G be a group and H be a characteristic subgroup of index two of G, then G_m^{n*} is contained in H, for any positive integer m and n.

Proof. The result is followed by lemma 2.4 [6]. □

3. n-IA-nilpotent groups

In the introduction, we introduced autonilpotent and G^{**}-autonilpotent groups. In this section, we define n-IA-nilpotent groups and study their properties.

Definition 3.1. A group G is called an n-IA-nilpotent group of class at most n if the series (1.2) finally terminate in the identity subgroup.

Remark 3.2. The autonilpotent groups are n-IA-nilpotent, but the converse is not generally valid. For example, all of the abelian groups are n-IA-nilpotent, but

$$G = \bigoplus_{i=1}^{l} \mathbb{Z}_{2^{m_i}}, \qquad l > 1, \; m_1 = m_2 \geqslant \cdots \geqslant m_l$$

is not autonilpotent, because $K_n(G) = G$.

Proposition 3.3. If G or H is not n-IA-nilpotent group, then $G \times H$ is not n-IA-nilpotent.

Proof. It is clear by lemma 2.2. □

Corollary 3.4. If $H_1, H_2, \ldots, H_l$ are n-IA-nilpotent groups with coprime orders, then $H_1 \times H_2 \times \cdots \times H_l$ is also n-IA-nilpotent.

Proof. The result is followed by induction on l. □

Theorem 3.5. Let G be a group and H be a characteristic subgroup of it. If H and G/H are n-IA-nilpotent, then G is also n-IA-nilpotent.

Proof. Because H and G/H are n-IA-nilpotent, so there exist positive integers i and j such that

$$H_i^{n*} = \langle 1 \rangle, \qquad \left(\frac{G}{H}\right)_j^{n*} = \langle 1 \rangle.$$

Clearly, $G^{n*}H/H = (G/H)^{n*}$ and by induction on j one gets

$$\frac{G_j^{n*}H}{H} \subseteq \left(\frac{G}{H}\right)_j^{n*} = 1_{\frac{G}{H}}.$$

So, $G_j^{n*} \subseteq H$. Let $[g, \alpha^n]$ be an arbitrary generator of $G_{j+1}^{n*} = [G_j^{n*}, IA(G)]$. It is not difficult to show that $[g, \alpha^n|_H] \in [H, IA(H)]$, thus $G_{j+1}^{n*} \subseteq H^{n*}$. By induction on i, we have $G_{i+j}^{n*} \subseteq H_i^{n*} = \langle 1 \rangle$. Therefore, G is an n-IA-nilpotent group of class at most i+j. □

Theorem 3.6. Let G be a group and H be a proper characteristic subgroup of it with G/H is n-IA-nilpotent of class c. If $H \cap G_c^{n*} = \langle 1 \rangle$, then G is n-IA-nilpotent.

Proof. Basing our argument in the proof of theorem 3.5, it is not difficult to show that $G_c^{n*}H/H \subseteq (G/H)_c^{n*}$. As $H \cap G_c^{n*} = \langle 1 \rangle$, we have

$$\frac{G_c^{n*}}{H \cap G_c^{n*}} \cong \frac{G_c^{n*}H}{H} \subseteq \left(\frac{G}{H}\right)_c^{n*} = 1_{\frac{G}{H}}.$$

Thus $G_c^{n*} = \langle 1 \rangle$ which yields the n-IA-nilpotency of G. □

Theorem 3.7. Let G be an n-IA-nilpotent group of class c and H be a characteristic subgroup H of it. If $G = HG^{n*}$, then G=H.

Proof. By hypothesis

$$G^{n*} = [G, IA(G)] = [HG^{n*}, IA(G)].$$

We prove that

$$[HG^{n*}, IA(G)] \leqslant [H, IA(G)]^{G^{n*}}[G^{n*}, IA(G)].$$

Assume that $h \in H$, $g \in G^{n*}$ and $\alpha \in IA(G)$, then $[hg, \alpha^n] \in [HG^{n*}, IA(G)]$ and

$$\begin{aligned}
[hg, \alpha^n] &= (hg)^{-1}\alpha^n(hg) \\
&= g^{-1}h^{-1}\alpha^n(h)\alpha^n(g) \\
&= g^{-1}h^{-1}\alpha^n(h)gg^{-1}\alpha^n(g) \\
&= [h, \alpha^n]^g[g, \alpha^n] \in [H, IA(G)]^{G^{n*}}[G^{n*}, IA(G)].
\end{aligned}$$

Because H is a characteristic subgroup, we have

$$\begin{aligned}
G^{n*} =& [HG^{n*}, IA(G)] \\
\leqslant& [H, IA(G)]^{G^{n*}}[G^{n*}, IA(G)] \\
\leqslant& HG_2^{n*}.
\end{aligned}$$

Thus

$$G = HG^{n*} \leqslant HG_2^{n*}.$$

So, $G = HG_2^{n*}$. Now, by induction, we have $G = HG_c^{n*}$, and G=H since G is an n-IA-nilpotent group of class c. □

References

[1] S. Bachmuth, Automorphisms of free metabelian groups, Trans. Amer. Math. Soc. 118(1965) (1963), 93–104. 1

[2] S. Barin, M. M. Nasrabadi, IA-Commutator Series, Paper presented at the 3rd National Congress on Mathematics and Statistics, Gonbad Kavous University, Golestan, 24 February 2022. (document), 1

[3] R. G. Ghumde, S. H. Ghate, IA-automorphisms of p-groups, finite polycyclic groups and other results, Matematicki Vesnik, 67 (2015), 194–200. 1

[4] P. V. Hegarty, The absolute centre of a group, J. Algebra, 169 (1994), 929–935. 1

[5] S. F. Mohebian, S. Hosseini, Some properties of n-autonilpotent groups, Asian-European Journal of Mathematics, 12(01) (2019). 1

[6] F. Parvaneh, M. R. R. Moghaddam, Some properties of autosoluble groups, J. Math. Ext. 5(1) (2010), 13–19. 1, 2